Solutions to Problems from 11/6 Handout

Exercise 1. *Prove that f is $1-1$ iff f^{-1} is a function from Y to X.*

Proof.

($\Longrightarrow$) Suppose that f is $1-1$. Then if $(x_1, y), (x_2, y) \in f$ we have that $x_1 = x_2$. Consider $(y, x_1), (y, x_2) \in f^{-1}$. We already know that $x_1 = x_2$. Thus, since x_1 and x_2 were arbitrary, f^{-1} is a function.

($\Longleftarrow$) Suppose that f^{-1} is a function. Then if $(y, x_1), (y, x_2) \in f^{-1}$, we have $x_1 = x_2$. Consider $(x_1, y), (x_2, y) \in f$. Since f^{-1} is a function, we have that $x_1 = x_2$. Thus, since x_1 and x_2 were arbitrary, f is a function.

□

Exercise 2. *Prove that if $A \subset B$ in X, then $f(A) \subset f(B)$ in Y.*

Proof. Let $f(a) \in f(A)$. Then $a \in A$. Since $A \subset B$, $a \in B$. Thus $f(a) \in f(B)$. Therefore, $f(A) \subset f(B)$. □

Exercise 3.

(1) *Prove or give a counterexample to the statement: $f(A \cap B) = f(A) \cap f(B)$.*
(2) *Prove or give a counterexample to the statement: $f(A \cup B) = f(A) \cup f(B)$.*

Proof.

(1) Let $A = (-1, 0)$ and let $B = (0, 1)$, then $A \cap B = \emptyset$. Let $f(x) = x^2$. Then $f(A \cap B) = \emptyset$, $f(A) = (0, 1)$, $f(B) = (0, 1)$, and $f(A) \cap f(B) = (0, 1) \neq \emptyset$. Thus $f(A \cap B) \neq f(A) \cap f(B)$.
(2) ($\subset$) Let $f(x) \in f(A \cup B)$. Then $x \in A \cup B$. So either $x \in A$ or $x \in B$. WLOG assume that $x \in A$. Then $f(x) \in f(A) \subset f(A) \cup f(B)$. Thus $f(A \cup B) \subset f(A) \cup f(B)$.
($\supset$) Let $f(x) \in f(A) \cup f(B)$. So either $f(x) \in f(A)$ or $f(x) \in f(B)$. WLOG assume that $f(x) \in f(A)$. Then $x \in A \subset A \cup B$. Then $f(x) \in f(A \cup B)$. Thus $f(A) \cup f(B) \subset f(A \cup B)$.

□

Exercise 4. *Prove that if $A \subset B$ in Y, then $f^{-1}(A) \subset f^{-1}(B)$ in X.*

Proof. This is done in a very similar way to exercise 2. □

Exercise 5.

(1) *Prove or give a counterexample to the statement: $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.*
(2) *Prove or give a counterexample to the statement: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.*

Proof.

(1) ($\subset$) We get $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$ from the previous exercise.
($\supset$) Let $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. Equivalently said, $f(x) \in A \cap B$. Thus $x \in f^{-1}(A \cap B)$.
(2) ($\supset$) We get $f^{-1}(A \cup B) \supset f^{-1}(A) \cup f^{-1}(B)$ from the previous exercise.
($\subset$) Let $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. WLOG, assume $f(x) \in A$. Thus $x \in f^{-1}(A) \subset f^{-1}(A) \cup f^{-1}(B)$.

□

Exercise 6. *Prove that the set of all finite subsets of* $\mathbb{N}$ *is countable.*

Proof. From a previous homework exercise we know that the subset $FS1 \subset 2^{\mathbb{N}}$ of sequences where all but finitely many terms are 0 is countable. We will create a bijection between this set and the subset $F \subset \wp(\mathbb{N})$. We have the function $\chi_A : X \to \mathbf{2} = \{0,1\}$ defined by $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$. Define $\chi : \wp(X) \to \mathbf{2}^X$ by $\chi(A) = \chi_A$. Let $X = \mathbb{N}$. To create this bijection we restrict the domain of χ to F. Let $S \in F$. Then S has a biggest number, call it n, since it is finite. Thus χ_S will have all zeros past the n^{th} term. Thus $\chi_S \in FS1$. Thus $\chi|_F : F \to FS1$. Let $q \in FS1$, then create a set, N, of numbers in the following way: if q has a 1 in the i^{th} position, then put i in N. Do this for each 1 in the sequence. Then, since there are finitely many $1's$, the set N will be finite and an element of F. Thus $\chi|_F$ is onto. Following the same idea, if we have that $\chi_C = \chi_D$, then we can see that $C = D$, so $\chi|_F$ is $1-1$. Therefore $\chi|_F$ is a bijection and thus F is countable. □